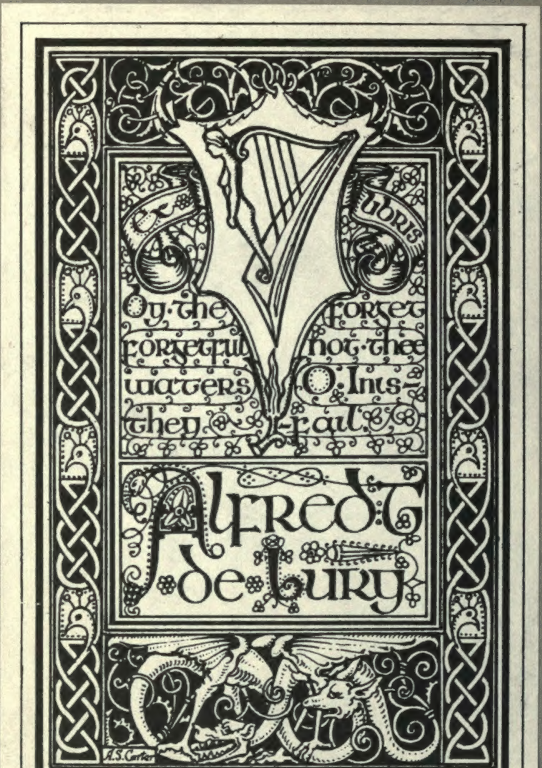
Loudon, William James
Notes on elementary
mechanics


Ex Libris
By·the forgetful waters they forget not·thee O·Inis-fail.
Alfred T. de Lury
A.S. Carter

University of Toronto

FACULTY OF ARTS

NOTES

ON

ELEMENTARY MECHANICS

BY

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NOTES ON
Elementary Mechanics

1. **Velocity** is rate of change of position. It involves the simple ideas of time and length, and has two elements, direction and magnitude.

2. **Uniform velocity**—when no force acts, or forces are in equilibrium. **Variable velocity,** when forces are not in equilibrium.

3. In uniform velocity, motion takes place in a straight line so that any two equal distances are passed over in equal times. If s is the distance, v the velocity, and t the time, then $s = vt$.

4. Newton's first law of motion is really a definition of uniform velocity. It states that a body's natural state is one of rest: that, if in motion, it will keep on moving in a straight line uniformly, until some external force changes its velocity either in direction or in magnitude or both.

5. If a wheel rolls steadily on a horizontal plane, then the *centre* moves with uniform velocity. If two weights are hung over a frictionless pulley by means of a fine string, and the system is disturbed, each point will move with uniform velocity. The centre of the sun may be moving in space with uniform velocity.

6. **Variable velocity.**

1. Direction constant
   Magnitude changing

Falling bodies, etc.

2. Direction changing.
   Magnitude constant

Circular motion.

3. Direction and magnitude both changing.
   1. Parabolic motion, under gravitation.
   2. Pendular and harmonic motion.

3. Wheel rolling along a horizontal plane. In this case, every point but the centre describes a cycloid, the velocity changing both in direction and magnitude.
4. Generally, all cases of motion in nature are of variable velocity, since it is impossible to get rid of natural forces, especially friction.

7. **Graphical representation of velocity.**

1. Falling body.... $v = gt$.
2. Body projected vertically upwards .... $v = u - gt$.
3. Path of a projectile.
4. Uniform circular motion.
5. Cycloid.

8. A body may have two or more velocities. Illustrated by the motion of a point on the earth.

Velocity of a point at the equator due to rotation

$$= \frac{2 \times \frac{22}{7} \times 4000}{24 \times 60 \times 60} = \tfrac{1}{4} \text{ mile a second.}$$

Velocity due to translation of the earth about the sun

$$= \frac{2 \times \frac{22}{7} \times 93{,}000{,}000}{365 \times 24 \times 60 \times 60}$$

$$= 20 \text{ miles a second.}$$

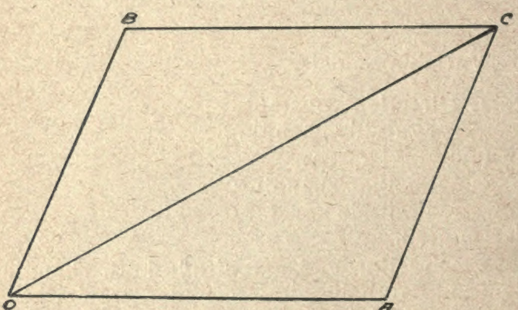

Fig. 1

Velocity of a point in latitude 45°, due to rotation

$$= \tfrac{1}{4} \times \cos 45^\circ = \frac{1}{4\sqrt{2}}$$

$$= \frac{7}{40} \text{ mile a second.}$$

9. Statement of the parallelogram of velocities. If, at any instant, a body has two velocities, represented graphically by OA, OB, then their resultant is represented in direction and magnitude by OC.

Illustration. A man rows across a stream, running uniformly between parallel banks.

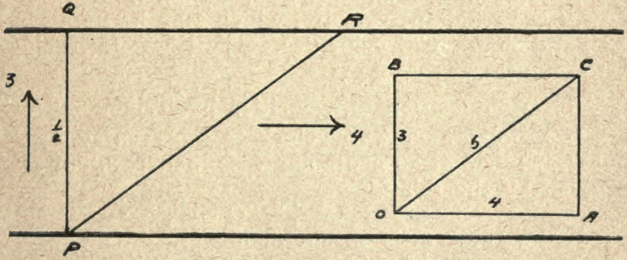

Fig. 2

Let velocity of stream be 4 miles an hour. Boat's velocity 3 miles an hour. And suppose the man to start from P and direct his boat always towards the other shore. Then he will evidently drift down stream and land at R. At each point of his path he will have two velocities represented by OA, OB. And his resultant will be $OC = 5$. Therefore his path is PR, parallel to OC. The time of crossing is independent of the velocity of the stream and is equal to PQ ($= \frac{1}{2}$) divided by 3, or 1-6 of an hour, ten minutes. The distance QR is equal to 1-6 of an hour, multiplied by 4, or 2-3 of a mile.

Illustration. Stream with variable velocity. Let the man row at the rate of 3 miles an hour, as in the previous case, but let the velocity of the stream vary from 0 at either side to 5 miles an hour at the centre of the stream.

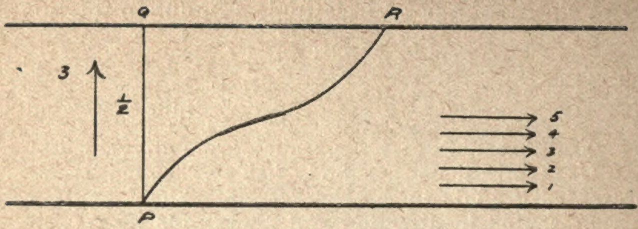

Fig. 3

Fig. 3 shows the path of the boat, which is curved. The line touching this curve at any point is the direction of the resultant velocity, which may be found by the parallelogram of velocities. The time of crossing is 10 minutes.

10. **Composition of velocities by Trigonometry.**

If a point have two velocities represented by P and Q, inclined at an acute angle, θ, their resultant, R, is given by the relation

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta$$

If the angle, θ, is obtuse then

$$R^2 = P^2 + Q^2 - 2PQ \cos \theta$$

If the angle θ is a right angle then

$$R^2 = P^2 + Q^2.$$

11. **Resolution of velocities.**

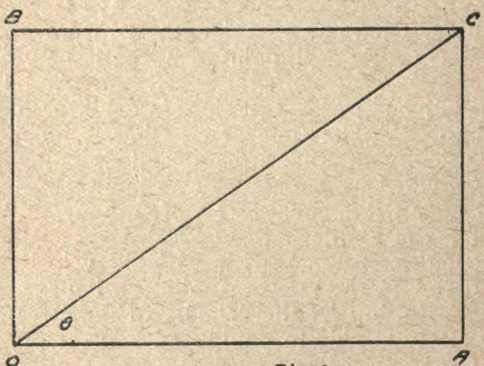

Fig. 4

It is evident that OC may be resolved into two *components:*

$$OA = OC \cos \theta$$
$$OB = OC \sin \theta$$

It is useful sometimes to consider these component velocities instead of the absolute velocity.

Path of a projectile.

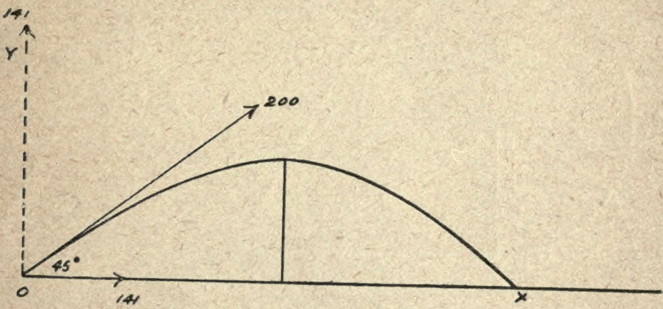

Fig. 5

Let a body be projected from a point O, with a velocity of 200 feet a second and at an angle of 45° to the horizontal line OX.

Then this velocity is equivalent to two velocities each equal to $\frac{200}{\sqrt{2}}$ or 141 feet a second in the directions shown. The horizontal velocity remains constant as the body moves, and the vertical velocity will be diminished by gravitation to the amount of g each second, according to the formula $v = u - gt$. If H represents the horizontal velocity, and V the vertical velocity, then

at end of 1″ $H = 141$; $V = 109$
at end of 2″ $H = 141$; $V = 77$
at end of 3″ $H = 141$; $V = 45$
at end of 4″ $H = 141$; $V = 13$
at end of 5″ $H = 141$; $V = -19$

When $v = o$, $t = \frac{141}{32} = 4\frac{13}{32}$ seconds.

This is the time the body will take to reach the

highest point. In twice this time it will reach the point X.

At each point of the path the tangent indicates the direction of the resultant or absolute velocity.

Hoop rolling on a horizontal plane.

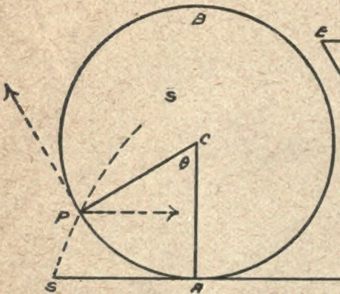

Fig. 6.

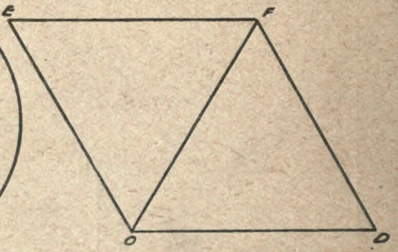

Fig. 7

Any point of the hoop will trace out a path called a cycloid. The absolute velocity is in the direction of the tangent to this curve, and is made up of two velocities, one of translation of the centre of the hoop, and the other of rotation about the centre considered as a fixed point.

The velocity of translation of the centre, C, is $a\omega$ where a is the radius of the hoop and ω is termed the *angular velocity,* or the angle described in a unit of time, expressed in circular measure: it is found by taking the time of a complete revolution and the distance traversed in that time.

The velocity of the highest point, B, of the hoop is $2a\omega$, and the velocity of the lowest point, A, is zero.

The velocity of any point P is found by compounding the two velocities of rotation and translation by the parallelogram of velocities, as shown in Fig. 7.

OD, OE represent the two velocities, parallel to the directions indicated by the arrows. They are each equal to $a\omega$, and the angle FDO is equal to θ or

PCA. Then the resultant velocity, OF, is given by the relation

$$OF^2 = OD^2 + DF^2 - 2OD.DF.\cos\theta$$
$$= a^2\omega^2 + a^2\omega^2 - 2a^2\omega^2\cos\theta$$
$$= 2a^2\omega^2(1 - \cos\theta) = 4a^2\omega^2\sin^2\frac{\theta}{2}$$

$$\therefore OF = 2a\omega\sin\frac{\theta}{2}.$$

P is therefore moving at the instant under consideration in a direction parallel to OF and at a rate equal to $2a\omega\sin\frac{\theta}{2}$. SPS' is a portion of the cycloidal path, described by the point P of the rolling hoop.

12. Acceleration of gravity.

(a) *Body falling from rest.*

If s be the distance passed over in any time t, then it is an experimental fact that $s = \frac{1}{2}gt^2$ where $g = 32$. The quantity g is termed the acceleration of gravity and means that the velocity of a falling body is increased at this rate (32 feet per second in one second). Consequently the velocity v of a falling body is given by the relation $v = gt$. Combining this with the previous relation we get $v^2 = 2gs$.

Formulae:
$$v = gt$$
$$s = \tfrac{1}{2}gt^2$$
$$v^2 = 2gs.$$

(b) *Body projected upwards.*

If a body be projected upwards with an initial velocity u, then in any time t, it would move through a distance ut, if gravitation did not act; but gravitation draws it back through a distance $\frac{1}{2}gt^2$; and therefore, in time t, it moves through an actual distance $ut - \frac{1}{2}gt^2$. Also, the velocity u is diminished every second by g: therefore the velocity at time t is $u - gt$. And since $s = ut - \frac{1}{2}gt^2$, and $v = u - gt$, therefore $v^2 = u^2 - 2gs$.

Formulae:
$$v = u - gt$$
$$s = ut - \tfrac{1}{2}gt^2$$
$$v^2 = u^2 - 2gs.$$

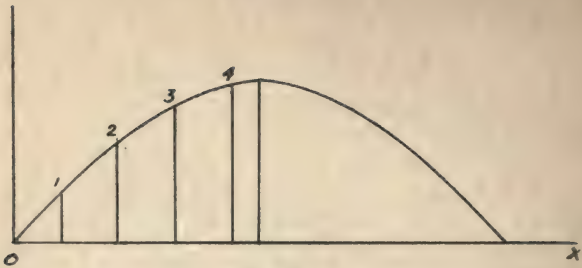

Fig. 8

13. Path of a projectile.

The actual path of a projectile is shown in Fig. 8. The body is supposed to be projected at an angle of 45° to the horizon, with a velocity of 200 feet a second. Its component horizontal and vertical velocities will be $\frac{200}{\sqrt{2}} = 100\sqrt{2} = 141$ feet a second, nearly. The body will therefore travel, in a horizontal direction, with a uniform velocity of 141 feet a second, and at the end of the first, second, third, and fourth seconds, will have travelled distances 141, 282, 423, 564 feet. The vertical velocity being continually diminished by gravitation, the distances passed over in the different intervals will be found from the relation

$$s = 141.t - \tfrac{1}{2}gt^2$$

The vertical distances it reaches at the end of first, second, third, fourth seconds will be, as calculated from this formula, 125, 218, 279, 308 feet. The coordinates of the points 1, 2, 3, 4 are therefore 141, 125; 282, 218; 423, 279; 564, 308; and these points lie on the curve shown, a parabola. The highest point reached will be found from the time to reach it, obtained from the relation

$$v = u - gt = 141 - gt$$

when $v = 0$, $t = \frac{141}{g} = \frac{141}{32} = 4\frac{13}{32}$ seconds.

The distance passed over in this time, the initial velocity being 141, is equal to $141 \times 4\frac{13}{32} - \tfrac{1}{2}g\left(4\frac{13}{32}\right)^2$

$= 310$ feet nearly; this is the greatest height. The total horizontal distance travelled is equal to $8\frac{13}{16} \times 141$.

14. **Experiments with falling bodies to prove the truth of the relation $s = \frac{1}{2}gt^2$.**

15. **Relative velocity.**

When two points are moving in the same plane the velocity of the first point relative to the second is found by reducing the second to rest, and imparting to the first the reverse velocity of the second, in addition to its own real velocity. For instance, when two trains are moving in opposite directions along parallel tracks, to a person in one train the other appears to be moving with its own real velocity together with the velocity of the second reversed. If one train be moving at the rate of 30 miles an hour and a second be moving in the opposite direction at the rate of 40 miles an hour, then, to a person in the first train, the second train will appear to move with a velocity of $40 + 30$, or 70 miles an hour.

If the trains moved in the same direction at the rate of 20 miles an hour, then, to a person on either one there would appear to be no motion of the other. If one moved at the rate of 30 miles an hour and the second at the rate of 20 miles an hour, both moving in the same direction, then to a person in the first train the second would appear to move backwards at the rate of 10 miles on hour, while, to a person in the second train, the first would appear to be advancing at the rate of 10 miles an hour.

In cases where the velocities are not in parallel directions, the relative velocity is obtained by means of the parallelogram law.

Illustrations.

1. A man walks north at the rate of 4 miles an hour, and a west wind is blowing at the rate of 10 miles an hour. Find the velocity of the wind relative to the man.

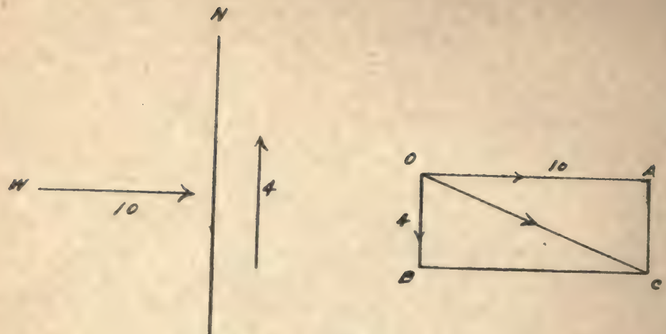

Fig. 9

Fig. 9 shows the solution. The man's velocity is 4 in a northerly direction, so that if he be reduced to rest the velocity of the wind will be found by the parallelogram of velocities: *OA* being the real velocity of the wind and *OB* the velocity due to the man being reduced to rest. The resultant velocity of the wind is *OC*, equal to $\sqrt{116}$. Consequently, the wind will appear to blow from a direction parallel to *OC*, at the rate of $\sqrt{116}$ miles an hour.

2. Smoke trail from a steamer, which is moving north at the rate of 10 miles an hour, while a west wind is blowing at the rate of 10 miles an hour.

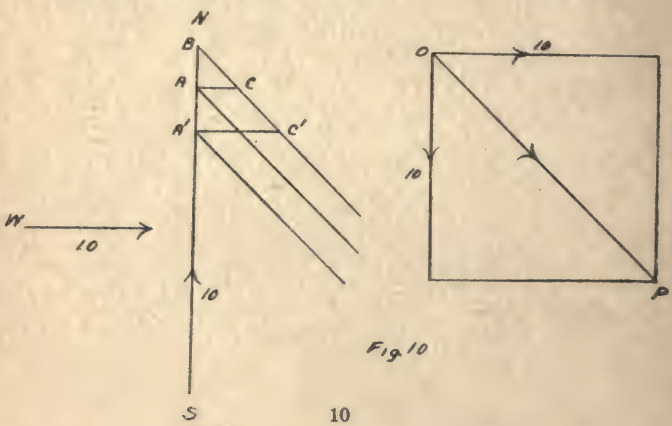

Fig. 10

The resultant velocity of the smoke particles relative to the steamer is shown by the diagonal *OP* of the parallelogram of velocities.

The trail of the smoke is always parallel to the line *OP*. Thus, while the steamer goes from *A* to *B* the smoke travels from *A* to *C*, and while the steamer goes from *A'* to *B*, the smoke particles go from *A'* to *C'*.

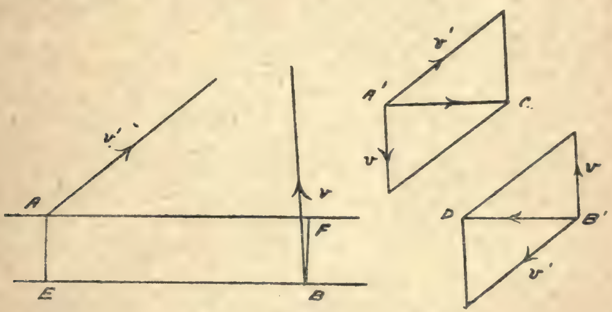

Fig. 11

3. General case of two points *A*, *B* moving with velocities v', v ($v' > v$).

The velocity of *A* relative to *B* is shown by *A'C*; and the velocity of *B* relative to *A* is shown by *B'D*. Viewed from *A*, the point *B* will appear to move along the line *BE*; viewed from *B*, the point *A* will appear to move along *AF*. Both *BE* and *AF* are parallel to the diagonals *A'C*, *B'D*.

The shortest distance between the two points as they move along is given by *AE*, or *BF*, drawn perpendicular to the common direction of the relative velocity *A'C* or *B'D*.

16. When we combine the idea of *mass* with that of acceleration or change of velocity, we get the idea of *Force*, which is measured usually by the product of a mass by an acceleration. Mass measured usually by weighing. Units of mass.

17. Force may be represented graphically by drawing a straight line in the proper direction and of the proper magnitude.

18. Forces:

Weight.
Tension of string.
Pressure.
Reaction.
Friction.

Other forces.

19. *One* force acting at a point produces motion. Newton's laws governing motion.

20. *Two* forces, combined by the parallelogram law similar to the parallelogram of velocities.

Also, a single force may be resolved into *components* by the parallelogram of forces: these components being determined either graphically or analytically by Trigonometry.

Illustration. Boat sailing by the action of wind.

In the case of a sail-boat the sail is supposed to be, theoretically, a plane surface: the nearer the approximation to this practically, the better the results.

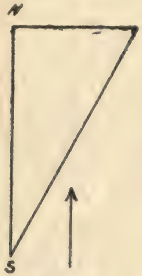

Fig. 12

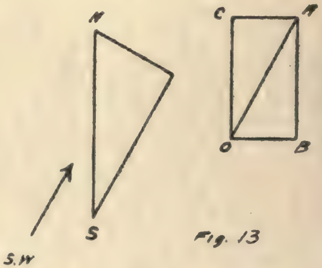

Fig. 13

In Fig. 12, a section of the sail is shown, the line of symmetry of the boat being represented by NS; and, the wind being south, the sail is set at right angles to the direction of motion, and the onward motion is produced directly by the action of the wind. In Fig. 13, the wind is south-west, and the sail is set as shown. In this case the effect of the wind is represented by OA, which, being resolved into two com-

ponents OB, OC, shows that the boat tends to move laterally, due to the component, OB. The boat therefore makes *leeway,* which is reduced more or less by a centreboard or keel. The onward motion is due to the component OC.

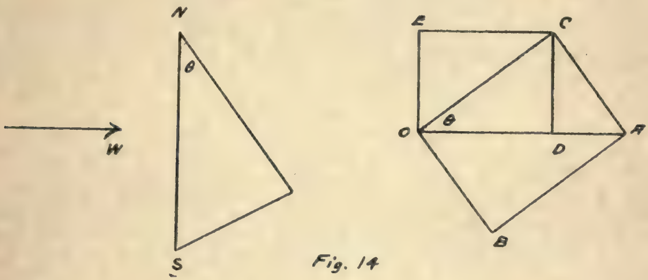

Fig. 14

Fig. 14 illustrates the action of a **beam** wind; and the boat travels in a direction at right angles to the wind, the sail being hauled in so as to make an acute angle with the line of symmetry of the boat. The total action of the wind is represented by OA drawn parallel to W. This is equivalent to two components OB, OC. And OC is equivalent to OD, OE. Thus the wind produces *leeway,* represented by OD, a tangential effect or *down-draught* represented by OB, and the onward motion is due to OE. If the angle which the sail makes with NS be θ, which is equal to COD, then

OA $(=W)$ is equivalent to $OB = W \sin\theta$ and $OC = W\cos\theta$. OC is equivalent to $OD = W\cos^2\theta$ and $OE = W\cos\theta\sin\theta$.

$$OB = W\sin\theta = \text{down-draught}$$

$$OD = W\cos^2\theta = \text{leeway}$$

$$OE = W\sin\theta\cos\theta = \text{onward motion.}$$

The best effect will be obtained when OE has its maximum value, that is when $\sin\theta\cos\theta$ is a maximum, which is true when $\theta = 45°$.

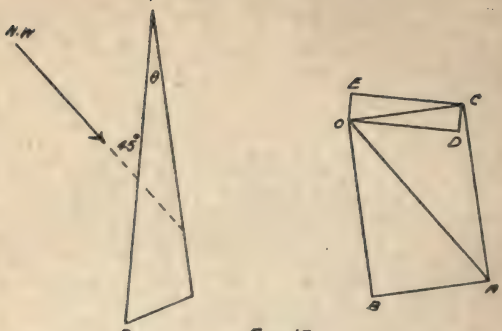

Fig. 15

Fig. 15 shows how a boat may make progress in a northerly direction even when the wind is north-west. The wind being represented by OA, this may be resolved into components OB, OD, OE, and if the boat be so constructed as not to make too much lee-way, then it will have an onward motion due to OE. In this case OA is parallel to N.W., OB is parallel to the sail, OD is perpendicular to the direction of motion, and OE is parallel to NS.

If the angle between the sail and NS be θ, then $BOA = 45^\circ - \theta$ and the components become:

$$OB = \cos (45 - \theta)$$
$$OD = \sin (45 - \theta) \cos \theta$$
$$OE = \sin (45 - \theta) \sin \theta$$

since the angle $COD = \theta$.

OE will be a maximum when $\sin (45 - \theta) \sin \theta$ is a maximum, or $2\theta = 45^\circ$.

21. **Three forces. Triangle of forces.**

If three forces, acting at a point, and in one plane, be in equilibrium, then they may be represented in direction and magnitude by the sides of a triangle taken in order.

This may be deduced from the parallelogram of forces, or proved by experiment, with a graduated circular board and weights over frictionless pulleys.

22. If three forces are not in equilibrium then

their resultant is determined graphically by the line which closes the broken figure.

23. Polygon of forces.

If any number of forces, acting at a point, and in one plane, be in equilibrium, then they may be represented in direction and magnitude by the sides of a closed figure, taken in order.

If they are not in equilibrium, their resultant is represented by the line which closes the broken figure. Illustrated experimentally.

24. Parallel forces.

Conditions of equilibrium:

1. The algebraic sum of the forces must be zero.
2. The algebraic sum of the moments taken about any point in the plane must be zero.

These conditions may be illustrated by means of a uniform beam, suspended freely, and loaded with different sets of weights.

(The *moment* of a force about a point is equal to the product of the number representing the force by the perpendicular distance from the point on the line of action of the force.)

25. Experimental illustrations of the laws of parallel forces.

Reactions of beam on two supports.

Beam resting with one end on table and supported by the tension of a string.

Beam resting partially on a table.

26. Centre of gravity, or Centroid.

The point at which the weight of a body may be supposed to act.

Assuming that the centre of gravity of a uniform rod is at the centre, it follows that the centre of gravity of any symmetrical body is at the centre of symmetry.

Also, if a body is symmetrical in respect of a line or plane, the centre of gravity lies in that line or plane.

27. **Triangular plate.**

The centroid may be obtained geometrically by dividing the plate into uniform thin slices. Or, it may be found experimentally.

28. The centroid of any thin plate found by experiment.

New definition of centroid: the point of a body about which rotation takes place most easily.

Any line passing through the centre of gravity of a body may therefore be looked upon as an axis of minimum effort.

Centre of gravity of two small spheres, joined together by a fine wire. Show experimentally that, if this system is suspended by a twisted thread, it will rotate about the centroid as a fixed point. Such a system in motion illustrates approximately the motion of the earth and sun.

29. **General method of finding the centre of gravity of a body or system of bodies.**

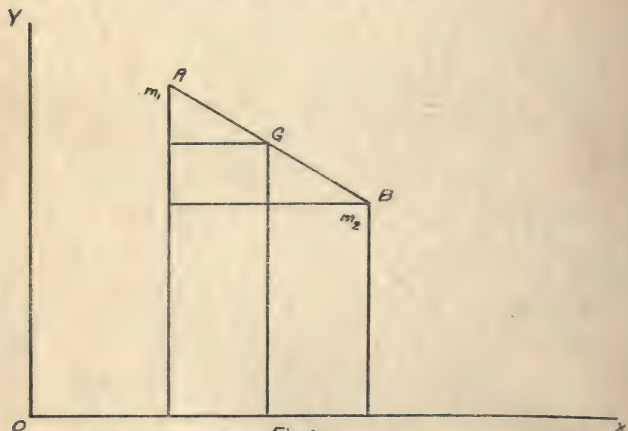

Fig. 16

Let two small masses, m_1, m_2, be at the points A, B: G, their centre of gravity is given by the relation

$$m_1 \times AG = m_2 \times BG.$$

Hence

$$m_1 (\overline{x} - x_1) = m_2 (x_2 - \overline{x})$$

the coordinates of A, B, G being, respectively, x_1, y_1; x_2, y_2; $\overline{x}$, $\overline{y}$.

Therefore

$$\overline{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Similarly, it may be shown that

$$\overline{y} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

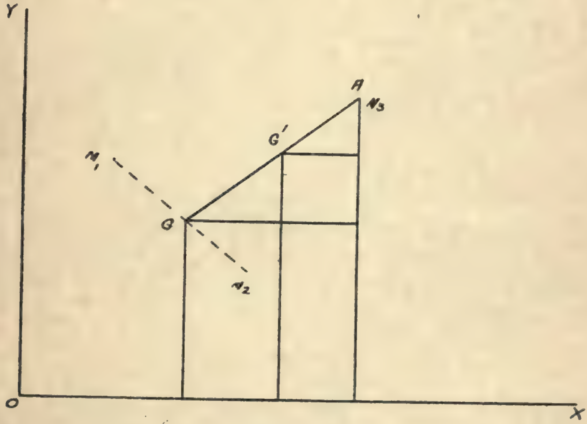

Fig. 17

If there are three small masses m_1, m_2, m_3 at the points x_1, y_1; x_2, y_2; x_3, y_3; then, replacing m_1 and m_2 by a mass $m_1 + m_2$ at G, G', the centre of gravity of the three masses m_1, m_2, m_3 will be defined by the relation

$$(m_1 + m_2)\, G\, G' = m_3 \times A\, G'$$

$$\therefore (m_1 + m_2)\left\{ \overline{x} - \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \right\} = m_3 (x_3 - \overline{x})$$

$$\therefore \overline{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

Similarly,

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

Therefore, generally, it may be shown that for number of masses m_1, m_2, ... m_n, the centre of gra is defined by the relations

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \ldots\ldots etc.}{m_1 + m_2 + \ldots\ldots etc.} = \frac{\Sigma\,(m\,x)}{\Sigma\,(m)}$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + \ldots\ldots + etc.}{m_1 + m_2 + \ldots\ldots + etc.} = \frac{\Sigma\,(m\,y)}{\Sigma\,(m)}$$

These expressions are true in the case w m_1, m_2, ... are infinitely small, and also in the where they represent finite masses collected at dividual centres of gravity.

They may also be applied where a portion body or several portions are removed. In such a one or more of the masses will be negative.

Illustrations.

1. To find the centre of gravity of a plat shown in Fig. 18: a square joined to an equilat triangle.

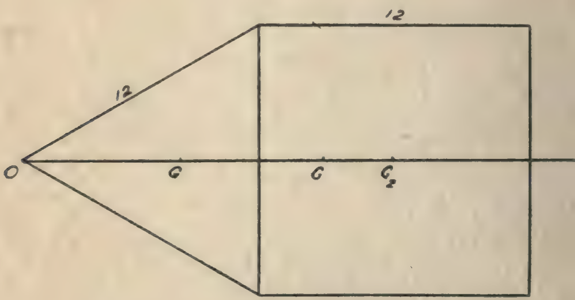

Fig. 18

The centre of gravity must lie on $O\,X$, the lin symmetry. Collect the mass of the triangle at G_1 the mass of the square at G_2.

Then

$$\bar{x} = OG_1 = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{36\sqrt{3} \times 4\sqrt{3} + 144 \times 6(1 + \sqrt{3})}{36\sqrt{3} + 144} = 13.4$$

2. A circular area of radius b is cut from a circular area of radius a, to find the centroid of the portion left.

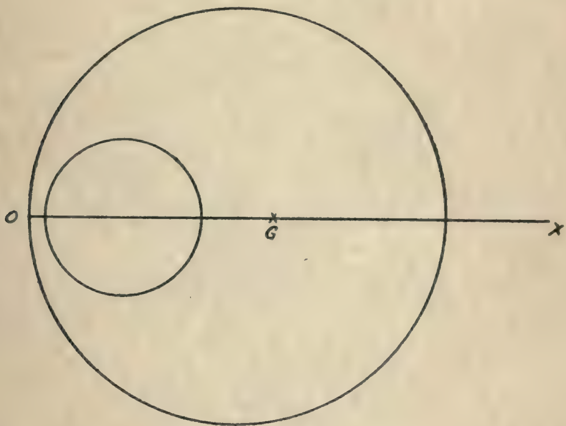

Fig. 19

If the radius of the larger circle be a, of the smaller one b, and if the distance between their centres be c, then the centroid required is at the point G, to the right of the centre of the larger circle, on the line of symmetry $O\ X$, such that

$$x = OG = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{\pi\ a^2.\ a - \pi\ b^2.\ (a - c)}{\pi\ (a^2 - b^2)}$$

$$= \frac{a^3 - ab^2 + b^2 c}{a^2 - b^2}$$

3. A semicircular arc, radius a.

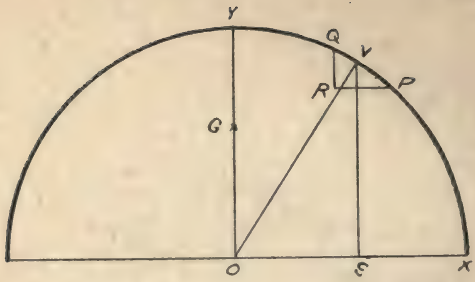

Fig. 20

In this case, the arc is divided into an infinite number of indefinitely small elements, such as PQ: and the formula will still apply.

Then, G will evidently be on the line of symmetry, $O\,Y$.

$$\text{and } y = OG = \frac{\Sigma(my)}{\Sigma(m)} = \frac{\Sigma(PQ \times y)}{\pi a}$$

considering the density of the arc as unity.

$$\text{But } \cos R\,P\,Q = \frac{P\,R}{P\,Q}$$

$$\cos S\,V\,O = \frac{S\,V}{V\,O}$$

V being the centre of PQ.

And the angles RPQ, SVO are equal.

$$\therefore \frac{P\,R}{P\,Q} = \frac{S\,V}{V\,O} = \frac{y}{a}$$

$$\therefore P\,Q = \frac{a}{y} \times P\,R$$

$$\therefore \bar{y} = \frac{\Sigma\left(y \times \frac{a}{y} \times P\,R\right)}{\pi a} = \frac{a\,\Sigma\,(P\,R)}{\pi a} = \frac{2\,a^2}{\pi a} = \frac{2\,a}{\pi}$$

30. Centroid of a weighted beam. Found by calculation and by experiment.

31. **Stability.** The stability of a body is dependent on the position of its centroid.

Illustrations of stable, unstable, and neutral equilibrium.

(a) Body, with an *area* of contact, resting on a horizontal plane.

(b) Body suspended from a fixed point.

(c) Body capable of motion about a fixed horizontal axis.

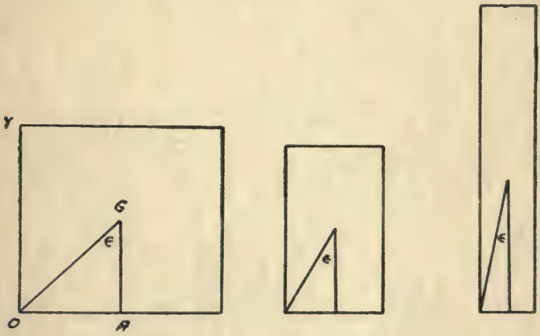

Fig. 21

32.
(a) In the case of a body which rests on a horizontal plane, with a comparatively large area of contact, the stability will depend on what is called the *angle of equilibrium.*

Thus, in Fig. 21, the angle of equilibrium is *OGA,* where *G* is the centroid, and *GA* is perpendicular to the area of contact. This means that the body may be turned about *O,* through the angle *VOG* or *OGA* before its equilibrium is destroyed. The smaller the area of contact, and the higher the centre of gravity, the smaller will become the angle of equilibrium.

Fig. 22 shows the angle of equilibrium of a right-angled triangular block, according as it is placed on its hypothenuse, or on one of its sides.

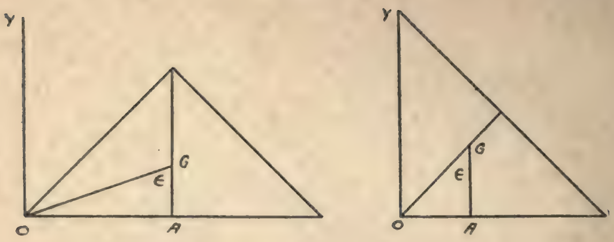

Fig. 22

(b) The pendulum.

Time of an oscillation is $\pi\sqrt{\frac{l}{g}}$: verified by experiment.

(c) The metronome and balance beam, illustrating the three conditions of stable, unstable and neutral equilibrium.

33. **The Chemical or Physical Balance.**

Two kinds of balance are in general use: those with *equal* arms and those with *unequal* arms. The theory of both depends on the principle of moments.

In the use of the balance the following points are to be remembered:

1. Levelling, to insure that, when in equilibrium, the beam is horizontal. Two levels of the ordinary type are generally used, in planes at right angles to one another, or else a single circular level.

2. Scalepans must be on knife edges and free to swing in all directions. This insures that the centre of gravity of the load will act in the vertical line of the knife edge.

3. The centre of gravity of the balance system should be below the axis of rotation, so as to produce stability.

4. In the final weighing a *rider* may be used, as it is not only more accurate but also more convenient than the use of very small weights. The use of the *rider* depends also on the principle of moments.

34. **Unequal armed balances.**

Theory explained by models.
Different types in use.

35. **Commercial balances.**

Steelyards.
Platform scales.

For weighing large quantities of material, the platform scale is generally used. It is made with a compound beam or set of beams, the principle of which is illustrated by Fig. 23.

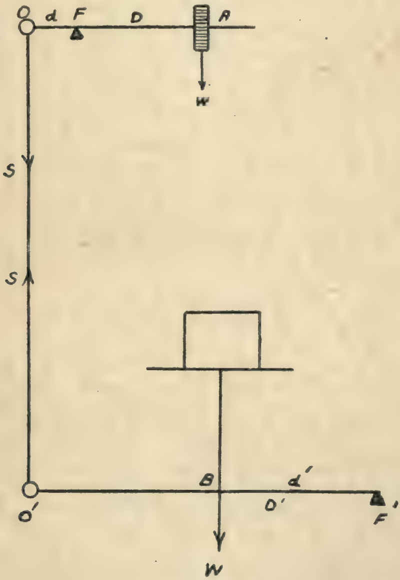

Fig. 23

The beam *OA* has a knife edge at *F* and a running weight *w*, which can be moved along the outer arm of the beam. At *O* there is a vertical rod, loosely

hinged, to the other end of which, at O', is attached a second beam with a knife edge at F'.

For equilibrium of w and W we have:

$$w \times D = S \times d \qquad (1)$$

where $OF = d$, $AF = D$, and S is the strain in the vertical rod.

Also $W \times d' = S \times D' \qquad (2)$

where $O'F' = D'$, $BF' = d'$, and W is the load to be weighed.

Combining (1) and (2) we get, by division,

$$\frac{uD}{Wd'} = \frac{D'}{d}, \text{ or } W = w\,\frac{DD'}{dd'}$$

Thus, by making D and D' large and d, d' small, we may weigh any large quantity W by means of a comparatively small weight w.

For example, if $d = d' =$ one inch,
$D = D' =$ ten inches,
$w =$ one pound,

Then $W = 100$ pounds.

36. **The use of the spring-balance.**

37. **The Lever.**

Any rigid rod, straight or curved, capable of motion about a fixed point called the *fulcrum.*

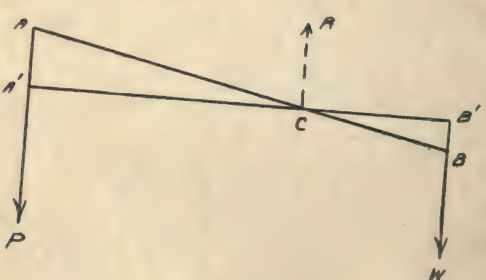

Fig. 24

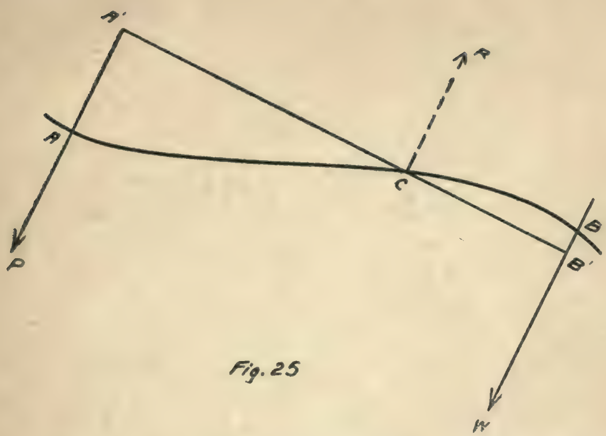

Fig. 25

The lever is represented by AB. C is the fulcrum. The power is applied at A, and is denoted by P. The resistance to be overcome is W, and the conditions of equilibrium are:

$$P \times A'c = W \times B'c$$
$$R = P + W$$

where R is the strain at the fulcrum.

Class I. Fulcrum between P and W.

Examples: Crowbar, scissors, shears, etc., beams of balances with equal or unequal arms.

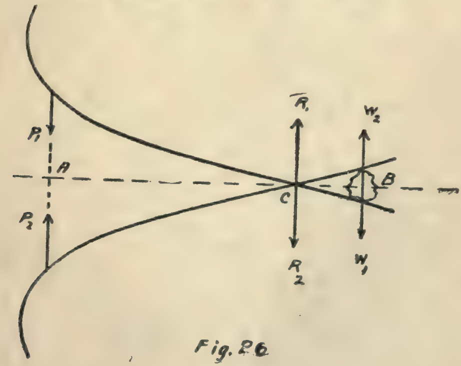

Fig. 26

Fig. 26 illustrates the action of a pair of scissors, consisting of two levers movable about a common fulcrum.

For instance, if $P_1 = 50$, $AC = 10$, $BC = \frac{1}{2}$
then $W_1 = 1000$, $R_1 = 1050$.

It should be noticed in the figure that the system $P_1\ R_1\ W_1$ form one set of forces in equilibrium; while $P_2\ R_2\ W_2$ form another set, also in equilibrium. And the strain on the point C consists of two forces, equal to one another, and acting in opposite directions; it is on this account that, when a pair of scissors is in constant use, there is a gradual loosening of the rivet at the intersection of the two levers.

Class II. Fulcrum beyond P and W, but nearer to W.

Examples: Nut-crackers, oars of a boat, crowbar, wheelbarrow.

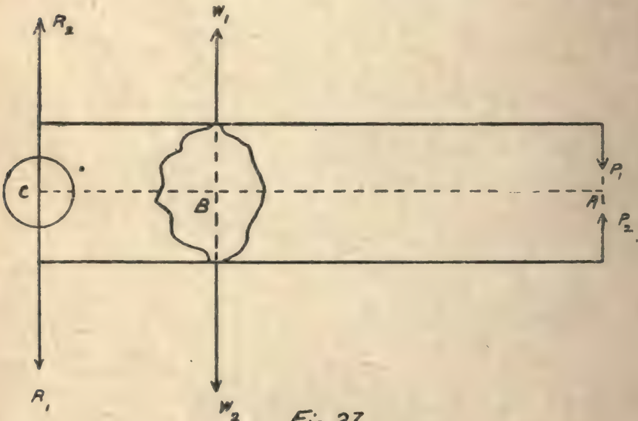

Fig. 27

Fig. 27 shows the forces brought into action when a pair of nut-crackers is used. There are two systems in equilibrium, $P_1\ R_1\ W_1$, and $P_2\ R_2\ W_2$. The strain on the axis is less than the resistance to be overcome.

Thus, if AC = six inches, BC = one inch, and $P_1 = P_2 = 10$ lbs., then

$$P_1 \times AC = W_1 \times BC$$
$$W_1 = 60 \text{ lbs.}$$
$$\text{and } R_1 = R_2 = 60 - 10 = 50 \text{ lbs.}$$

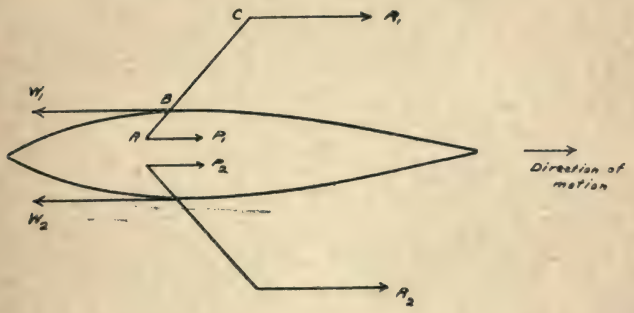

Fig. 28

Fig. 28 illustrates the action of the oars of a boat, when in use.

$$P_1 \times AC = W_1 \times BC$$
$$R_1 = W_1 - P_1.$$

The fulcrum, C, is in motion. Thus, if $P_1 = 250$ lbs., $AC = 8$ ft., $BC = 6$ ft., then $W_1 = 333\frac{1}{3}$, and $R_1 = 83\frac{1}{3}$.

Class III. Fulcrum beyond P and W, but nearer to P.

Examples: Human forearm, pliers, tongs, crane.

Levers of this class are non-efficient, since P is always greater than W. They are used for convenience.

38. **Three forces acting on a rigid body.**

If three forces act on a body and keep it in equilibrium, then they must either meet in a point or be parallel to one another.

Examples:

1. Beam resting on a table, lifted at one end.

2. Beam resting between two pegs.

3. Rod, suspended from a frictionless pulley, by means of a fine string attached to both ends.

Also, from a rough peg.

4. Equilibrium of a door, swung on two hinges.

5. Forces which keep a kite in equilibrium.

6. Force necessary to draw a wheel over an obstacle of given height.

7. A plate, in the form of an equilateral triangle, is suspended by a string attached to one of its angular points: to find the horizontal force, which, if applied at one of the other angular points, will keep it in equilibrium with one side vertical.

39. **Friction.** Sliding and rolling friction. One general law for both, that *friction always tends to oppose motion.*

40. **Laws of sliding friction.**

Illustrated by horizontal plane and inclined plane.
Coefficients of friction, and angle of friction.

41. **Friction of a rope around a post.**

Explained experimentally.

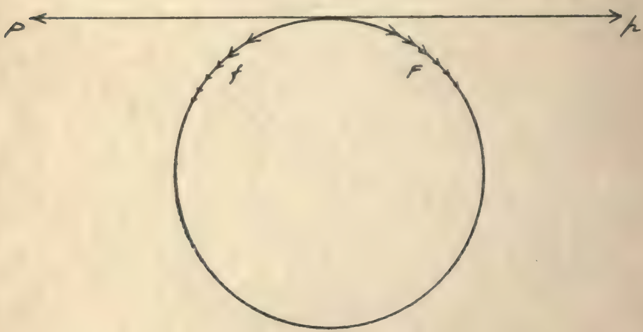

Fig. 29

Figs. 29 and 30 show the action of friction, when a cord is wrapped around a post.

Fig. 29 illustrates static equilibrium, just before motion begins; and we have

$$P + f = p + F.$$

P being greater than p, F will be greater than f. The forces of friction do not extend much beyond the point of contact of the two portions of the cord.

In Fig. 30, the system is in motion in the direc-

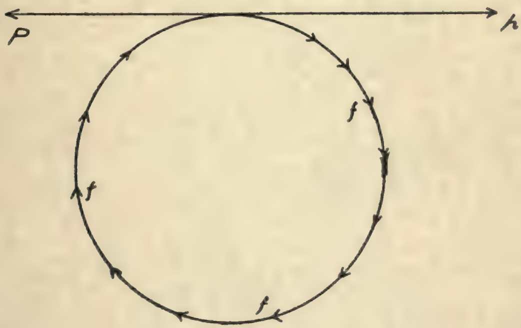

Fig. 30

tion of P, and the whole cord is acted on by friction forces as shown.

42. **Coplanar forces.**

Conditions of equilibrium:

1. The algebraic sum of the components in any direction must be zero.

2. The algebraic sum of the components in a perpendicular direction must be zero.

3. The algebraic sum of the moments of the forces about any point in the plane must be zero.

These are deduced from the equilibrium of parallel forces.

43. The principle of *work* and *energy*.

EXAMPLES.

VELOCITY

1. Find direction and magnitude of resultant of two equal velocities 6 feet per second, inclined at an angle of 120°.

2. Find magnitude of velocities 4 and 10 inclined at an angle of 60° to each other.

3. A particle is subject to two velocities, 10 and 15; what is the greatest and least velocity the particle may have? and how must the two component velocities be arranged respectively?

4. $\sqrt{37}$ is the resultant of the two velocities 3 and 4: at what angle are those velocities inclined to each other?

5. Find the resultant of two velocities 4 and 8 inclined to each other at the angle cos -1¾.

6. A particle has two simultaneous velocities of 12 feet, and 7 feet per second; inclined to each other at an angle of 45°: what is the direction of the resultant velocity?

7. Find the resolved part of a velocity of 15 feet per second in a direction, making an angle of 60° with the given direction.

8. A train is moving with a velocity of 30 miles per hour in a north-easterly direction: what is the northerly component of that velocity?

9. A velocity of $6\sqrt{2}$ along the diagonal of a square is resolved into two rectangular components along the sides of the square. How much is each component?

10. A velocity of 13 is resolved into rectangular components, the one making an angle cos -1 5/12 with the original velocity: what are the component velocities?

11. A ball, moving at the rate of 10 feet per second, is struck in such a way that its velocity is increased to 12 feet a second, and the direction of the new velocity makes an angle of 45° with that of the old velocity; find by construction the velocity imparted by the blow and its direction.

12. A person in a train moving 60 miles per hour wishes to hit a stationary object which is situated 100 yards off in a line through the marksman at right angles to the line of motion of the train. If his bullet moves 1200 feet per second, find out how much to one side of the object he should aim?

13. In a current which flows to the E. at the rate of $2\frac{1}{2}$ miles an hour, a steamer is going with its head 15° to the N. of N.E. at the rate of 8 miles an hour; find the true velocity of the steamer.

14. To a passenger in a train, raindrops seem to be falling at an angle of 30° to the vertical: they are really falling vertically, with velocity 80 feet per second. What is the rate of the train?

15. Two roads cross at right angles, along one a man walks northwards at 4 miles per hour, along the other a carriage goes at 8 miles per hour: what is the velocity of the man relative to the carriage?

16. Two trains start at the same time from the same station, and move along straight lines of railway, which are at right angles to each other, at the rates of 25 and 60 miles an hour respectively: find their relative velocity.

17. Two trains, whose lengths respectively were 130 feet and 110 feet, moving in opposite directions on parallel rails, were observed to be 4 seconds in completely passing each other, the velocity of the longer train being double that of the other: find at what rate per hour each train is moving.

18. The velocity of the extremity of the minute-hand of a clock is 48 times the velocity of the extremity of the hour-hand, which is 3 inches long: find the length of the minute-hand.

19. One ship sailing east with a speed of 15 knots passes a certain point at noon. A second ship sailing north at the same speed passes the same point at 1.30 p.m. At what time are they closest together, and what is then the distance between them?

20. Two steamers, X and Y, are respectively at points A and B, 5 miles apart. X steams away with a uniform velocity of 10 miles an hour, making an angle of 60° with AB: find in what direction Y must start

at the same moment, if it steam with a uniform velocity of $10\sqrt{3}$ miles per hour, in order that it may come into collision with X, and at what angle will it strike it?

ACCELERATION OF GRAVITY

1. What velocity has a body after falling freely for one minute?

2. How far does a stone fall during the third second of its motion?

3. A stone is thrown vertically upwards and returns to the point of projection after nine seconds: what was the initial velocity and the greatest height?

4. Determine the height required for a body to fall to get a velocity of a mile a minute; and the velocity due to the fall of 1 mile.

5. From what height must a body fall in order that it may have a velocity of 50 metres per second on striking the ground?

6. A stone is thrown down from the top of a tower 45 feet high, with a velocity of 6 feet per second: when will it reach the ground? What does the negative root of the equation indicate?

7. A tower is 288 feet high; at the same instant one body is dropped from the top of the tower and another projected vertically upwards from the bottom, and they meet half way: find the initial velocity of the projected body and its velocity when it meets the descending body.

TRIANGLE OF FORCES

1. A weight of 1 cwt. rests on an inclined plane of inclination $30°$: what are the resolved parts along and perpendicular to the plane?

2. A horse is attached by a chain to a railway wagon, the chain is inclined at an angle of $45°$ to the rails, the force exerted by the horse is 6 cwts.: what is the effective force along the rails?

3. Three equal forces act in one plane, on a point, in such a way that each of them makes an angle of $120°$ with each of the other two; prove that the forces will balance.

4. What is the angle between the forces 5 and

12 lbs., that they may be in equilibrium with a force of 13 lbs. when all three act on a particle?

5. Three forces acting at a point are in equilibrium; the greatest and least are 10 lbs. and 6 lbs., and the angle between two of the forces is a right angle: find the other force.

6. Two strings, one of which is horizontal, and the other inclined to the vertical at an angle of 30°, support a weight of 10 lbs.: find the tension in each.

7. Two given forces meet at a point: find in what direction a third force of given magnitude must act at the point, if the resultant of the three is the greatest possible.

8. Three equal forces, *P*, diverge from a point, the middle one being inclined at an angle of 60° to each of the others: find the resultant of the three.

9. Draw a diagram, showing the forces which keep a kite in equilibrium.

POLYGON OF FORCES

1. Forces of 9 lbs., 12 lbs., 13 lbs., 26 lbs. act on a point so that the angles between the successive forces are equal; find their resultant.

2. Forces whose magnitudes are proportional to the numbers 1, 2, 3, 4, 5, 6, 7, 8 act at a point in direction N., N.E., E., S.E., S., S.W., W., N.W. respectively: find the magnitude and direction of their resultant.

3. A number of forces act in one plane on a particle. Express geometrically the condition they must fulfil that they may maintain equilibrium. If this condition does not obtain, show how to determine geometrically their resultant.

4. If the forces acting on a point be represented by the sides of a regular pentagon taken in order, show at what angle the forces acting on the point are inclined to each other; and if the forces be so taken as acting on a point, prove independently of the polygon of forces that their resultant is zero.

5. Show that if a particle, placed in the centre of a regular polygon, be acted on by forces represented by the lines drawn from the particle to each of the angles it will be at rest.

6. Four forces act at a point and parallel to the sides of the rectangle *ABCD,* and are measured by those sides; the first three, *AB, BC, CD* act in a contrary sense to the fourth, *AD*: find their resultant and its line of action.

PARALLEL FORCES

1. Two men, one stronger than the other, have to remove a block of stone weighing 300 lbs. with a light plank whose length is 6 feet; the weaker man cannot carry more than 100 lbs.: how must the stone be placed on the plank so as to just allow him that share of the weight?

2. A horizontal rod without weight, 6 feet long, rests on two supports at its extremities; a weight of 6 cwt. is suspended from the rod at a distance of 2½ feet from one end: find the reaction at each point of support. If one support could only bear a pressure of 1 cwt., what is the greatest distance from the other support at which the weight could be suspended?

3. The horizontal roadway of a bridge is 30 feet long and its weight, 6 tons, may be supposed to act at its middle point, and it rests on similar supports on its ends: what pressure is borne by each of the supports when a carriage weighing 2 tons is one-third of the way across the bridge?

4. Three parallel forces act on a horizontal bar. Each = 1 lb. The right hand one acts vertically upwards, the two others verticall ydownwards, at distances 2 feet and 3 feet respectively from the first: draw their resultant, and state exactly its magnitude and position.

5. Forces of 4, 8, 6, and 10 lbs. act vertically downwards at equal distances along a horizontal line: what is the point of application of their resultant?

6. A light rod, 16 inches long, rests on two pegs, 9 inches apart, with its centre midway between them. The greatest weights which can be suspended from the two ends of the rod without disturbing the equilibrium are 4 lbs. and 5 lbs. respectively. There is another weight fixed to the rod: find that weight and its position.

7. If a light rod, ABC, is supported in a horizontal position by being placed under a peg at A, and over a peg at B: find the reactions of the pegs due to hanging a weight, W, at C.

8. A rod of uniform section and density, weighing 3 lbs., rests on two points, one under each end; a movable weight of 4 lbs. is placed on it: where must this weight be placed that one of the points may sustain a pressure of 3 lbs., and the other a pressure of 4 lbs.?

CENTRE OF GRAVITY

1. Mention an experimental way of showing that the centre of gravity of a circular board is at its centre.

2. A body is in shape a sphere, but loaded in such a manner that its centre of gravity is not at its geometrical centre: when it is placed on a horizontal plane, what are its positions of stable and unstable equilibrium?

3. Two rods of uniform density are put together so that one stands on the middle point of the other and at right angles to it; the former weighs 3 lbs. and the latter 2 lbs.: find the centre of gravity of the whole.

4. Equal weights are placed at the angular points of a regular pentagon: show that their centre of gravity is at the centre of the circumscribing circle.

5. Weights of 2, 3, 2, 6, 9, 6 kilogrammes are placed at the angular points of a regular hexagon, taken in order: determine the position of their centre of gravity.

6. A rod (AB) of given length consists of two parts of equal cross section joined end to end, the specific gravity of the part towards A is to that of the part towards B, as 1 to 5: if the centre of gravity of the whole is two-thirds of the length of the rod from A, what part of the length is the heavier material?

7. A triangular piece of paper is folded across the line bisecting two sides, the vertex being thus brought to lie on the base: find the centre of gravity of the paper in this position.

8. A cross is formed of six equal squares joined together: find the position of the centroid.

9. A circular lamina, 4 inches in diameter, has two holes cut out of it, one 1.5 inches in diameter and the other 1 inch in diameter, with their centres 1 inch and 1.25 inches, respectively, from the centre of the lamina, and situated on diameters mutually perpendicular. Find the centre of gravity of the portion left.

Ans: 0.202 inches from the centre.

10. If a right angled triangular lamina be suspended from one of the points of trisection of the hypothenuse, show that it will rest with one side horizontal.

FRICTION

1. Define the coefficients of friction.

A weight of 500 lbs. is placed on a table, and is just not made to slide by a horizontal pull of 155 lbs.; find the coefficient of friction, and the number of degrees in the angle of friction by drawing it to scale; or, if you have no instruments, explain how to calculate the number of degrees.

2. A cube is placed with one edge on a rough horizontal plane (coefficient of friction $= \mu$), and a parallel edge on a smooth plane, inclined at an angle of $45°$ to the horizon: if θ is the inclination of the base of the cube when in the position in which it will just not slide into a lower position, show that

$$(1 + 3\mu) \tan \theta = 1 - \mu$$

3. A block of wood, of length a and height b, is placed on an inclined plane, with its length along the line of greatest slope; the inclination of the plane is gradually increased: show that the block will slide before toppling over, if $a > \mu b$.

4. A uniform heavy rod, AB, rests on the ground at B, and against a smooth vertical wall at A. If a stretched string without weight join C, the middle point of AB to D, the foot of the perpendicular from B on the wall, give a diagram showing the forces which act on AB. If the ground is smooth, could AB remain in equilibrium? Give a reason for your answer.

5. A body of known weight is placed on a rough

horizontal plane, and pulled in a certain direction: find (1) the force of the pull which will just make the body slide; and (2) what must be the direction of the pull that it may be the least that will make the body slide?

Suppose that P is the least pull as above determined, and suppose that the body when pushed by a force P_1 (acting along the same line, but in opposite direction, as the force P acted) is on the point of sliding, show that

$$P_1 (1 - \mu^2) = P (1 + \mu^2).$$

6. A ladder of length 30 feet, supposed to be without weight, rests with one edge on rough ground ($\mu = .4$) and the other on a smooth inclined plane of angle 45°; a man walks up the ladder: find how far he can go before the ladder slips, its inclination being $\tan^{-1}.4$.

CO-PLANAR FORCES

1. A rod, AB, is hinged at A, and supported in a horizontal position by a string, BC, making an angle of 45° with the rod; the rod has a weight of 10 lbs,. suspended from B: find the tension in the string and the force at the hinge. The weight of the rod may be neglected.

2. A pole 12 feet long, weighing 25 lbs., rests with one end against the foot of a wall, and from a point 2 feet from the other end a cord runs horizontally to a point in the wall 8 feet from the ground: find the tension of the cord and the pressure of the lower end of the pole.

3. An ordinary rectangular door, being supposed supported, in the usual manner, by two hinges, A and B, in a vertical line not passing through its centre of gravity, C: determine, given all particulars, the horizontal pressures H and K on the hinges arising from the weight W of the door.

4. A uniform beam, 12 feet in length, rests with one end against the base of a wall which is 20 feet high: if the beam be supported by a rope, 13 feet long, attached to the top of the beam and to the summit of the wall, find the tension of the rope, neglecting

its weight, and assuming the weight of the beam to be 100 lbs.

5. A light smooth stick 3 feet long is loaded at one end with 8 ozs. of lead; the other end rests against a smooth vertical wall, and across a nail which is one foot from the wall: find the position of equilibrium and the pressure on the nail and on the wall.

MISCELLANEOUS

1. Compare the angular velocities of the hour-hand, the minute-hand, and the second-hand of a watch.

2. If the earth's radius be 4000 miles, find the linear velocity of a point situated on the equator in foot-second units.

3. Compare the linear velocities of the extremities of the hands of a clock, the hour-hand being 2.2 inches long, and the minute-hand 3.6 inches.

4. A wind is on the beam; as the ship's speed increases the wind appears to draw more ahead. Explain this by a diagram.

5. A stone falls freely for 3 seconds when it breaks a pane of glass, and thereby loses one-half its velocity: find the height of the glass above the ground if the stone reaches the ground in 2 seconds after breaking the glass.

6. A body is thrown up with a velocity of 100 f.-s., and after 1 second another is thrown up and overtakes the other when at its highest point; find the velocity with which the latter is projected.

7. One ship sailing due N. is 6 miles south of another sailing twice as fast due E.: what will be the shortest distance between them, and when will the bearing of one from the other be N.E.?

8. A point has moved from rest with a uniform acceleration, and at the end of t_1, t_2 seconds its velocities are v_1 and v_2 respectively; find the space described in t seconds.

9. Two candlesticks, one six inches and the other a foot high, stand on a table two feet apart, and hold candles each a foot long at the moment when lighted. The candles burn at the rate of 1/20 inch per

minute. Find (1) the velocity of the shadow of the top of one candle thrown on the table by the other, and (2) the average velocity of the shadow of the top of the lower candlestick during the whole time which the candles take in burning away.

10. Forces $P - Q$, P, $P + Q$, act at a point in directions parallel to the sides of an equilateral triangle taken in order; find their resultant.

11. Show that a body has one, and only one, centre of gravity.

12. If a point be kept in equilibrium by forces represented in magnitude and direction by OP, OQ, OR, show that O is the C.G. of the triangle PQR.

13. A string of length $7a$ is fastened to the ends of a uniform rod, whose length is a, and passes over six smooth tacks placed so that the rod hangs in a horizontal position, and the whole forms a regular octagon; find the pressure on each of the tacks, and the tension of the string.

14. Three pegs in a vertical wall are placed at the angles of an equilateral triangle, the two lower ones being in a horizontal line. A string passing over the three pegs supports a body weighing 5 lbs. at each end. Find the tension of the string, and the pressure on each peg.

15. A and B are two hooks, in the same horizontal line, to which the ends of a cord 15 feet long are attached. If a body weighing 130 lbs. hang from the middle of the cord, and the cord can bear the strain of 140 lbs. weight, find the greatest distance between A and B consistent with the safety of the cord.

16. A dealer has correct "weights," but one arm of his balance is longer than the other. He sells an apparent weight W twice, using first one pan and then the other; what does he gain or lose?

17. A body, whose weight is 50 lbs., rests by friction only on a plane whose inclination is 30°; find the force of friction in action, and also the pressure exerted on the plane.

18. A uniform bar 3 feet long and weighing 5 lbs. rests on a horizontal table with one end projecting

4 inches over the edge; find the greatest weight that can be hung on the end without making the bar topple over.

19. A wheel 4 feet in diameter rolls uniformly along a horizontal plane, making 3 revolutions in 4 seconds: find the velocity of its centre, top, bottom, at any instant.

20. A wheel 6 feet in diameter rolls steadily along a horizontal road, making one revolution in two seconds: find the velocity of a point which, at any instant, is 4 feet from the ground.